# The Incomplete Gamma Function
# Part IV - A Mean-Reverting, Return Model

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The base equation for a mean-reverting process from Part II where the variable $t$ is time in years is... [1]

$$f(t) = \int\limits_m^n \mathrm{Exp}\left\{d + c\,t - a\,\mathrm{Exp}\left\{-\,b\,t\right\}\right\}\delta t \;\;\text{where...}\;\; a > 0\;,\; b > 0\;,\; c < 0\;,\; n > m \geq 0 \tag{1}$$

The solution to the base equation where $\Gamma(x, y)$ is the incomplete gamma function is... [1]

$$f(t) = \mathrm{Exp}\left\{d\right\} a^{\frac{c}{b}} b^{-1}\left[\Gamma\left(-\frac{c}{b}, a\,\mathrm{Exp}\left\{-\,b\,n\right\}\right) - \Gamma\left(-\frac{c}{b}, a\,\mathrm{Exp}\left\{-\,b\,m\right\}\right)\right] \tag{2}$$

In Part IV of the series on the incomplete gamma function we will build a return model that incorporates mean reversion. To that end we will use the following hypothetical problem...

## Our Hypothetical Problem

We are tasked with valuing a company given the following model parameters...

**Table 1: Valuation Model Assumptions**

| Description | Value |
|---|---:|
| Current annualized revenue (in dollars) | 1,000,000 |
| Current revenue growth rate (%) | 18.00 |
| Current return on assets (%) | 15.00 |
| Assets to annualized revenue (%) | 80.00 |
| Cost of capital (%) | 10.00 |

The projected annual growth rate of nominal GDP over the next ten years is expected to be four percent. Assume that the mean-reversion half life is five years. We will use our model to answer the following questions:

**Question:** What is the value of this company?

## Mean Reversion

We are currently standing at time zero where the current rate is unsustainable as market forces will either increase or decrease that rate to the long-term sustainable mean over time. We will make the following definitions...

| Term | Description |
|---|---|
| current rate | Market rate at time zero (unsustainable rate) |
| long-term rate | Market rate at time infinity (long-term sustainable mean) |
| lambda | The rate of mean reversion where $0 < \lambda < 1$ |

Using the definitions in the table above we will define the market rate at some future time $t$ to be the following equation where the variable $t$ is time in years...

$$\text{market rate at time } t = \text{long-term rate} + \left(\text{current rate} - \text{long-term rate}\right) \times \mathrm{Exp}\left\{-\,\lambda \times t\right\} \tag{3}$$

Using Equation (3) above note that at time zero the market rate is...

$$\text{market rate} = \text{long-term rate} + \left(\text{current rate} - \text{long-term rate}\right) \times \text{Exp}\left\{ -\lambda \times 0 \right\} = \text{current rate} \tag{4}$$

Using Equation (3) above note that at time infinity the market rate is...

$$\lim_{t \to \infty} \text{market rate} = \text{long-term rate} \;\; \text{...because...} \;\; \lim_{t \to \infty} \text{Exp}\left\{ -\lambda\, t \right\} = 0 \;\; \text{...if...} \;\; \lambda > 0 \tag{5}$$

In the equations above we defined the variable $\lambda$ to be the rate of mean reversion. To calibrate $\lambda$ we will choose some future point in time (time $= T$) where the market rate is halfway between the rate at time zero and the rate at time infinity (i.e. the half life). The equation to calibrate $\lambda$ is therefore...

$$\text{Exp}\left\{ -\lambda \times T \right\} = 0.50 \;\; \text{...such that...} \;\; \lambda = -\frac{\ln(0.50)}{T} \tag{6}$$

## Revenue

We will define the variable ARGR to be the discrete time annualized revenue growth rate, which is defined as the year over year growth rate of revenue. The generalized equation for the annualized revenue growth rate in discrete time is...

$$\text{ARGR} = \text{Cumulative revenue for the current year} \div \text{Cumulative revenue from the prior year} - 1 \tag{7}$$

We will define the variable $\mu_t$ to be the continuous time revenue growth rate at time $t$. The revenue growth rate at time $t$ is a function of the variable $\omega$, which is the long-term sustainable rate, the variable $\Delta$, which is the difference between the current unsustainable rate and the long-term sustainable rate, the variable $\lambda$, which is the rate of mean reversion, and the variable $t$, which is time in years. The equation for the revenue growth rate in continuous time is...

$$\mu_t = \omega + \Delta\, \text{Exp}\left\{ -\lambda\, t \right\} \;\; \text{...where...} \;\; \omega = \mu_\infty \;\; \text{...and...} \;\; \Delta = \mu_\infty - \mu_0 \tag{8}$$

We will define the variable $\Gamma_t$ to be the cumulative revenue growth rate realized over the time interval $[0, t]$. Using Equation (8) above the equation for the cumulative revenue growth rate at time $t$ is...

$$\Gamma_t = \int_0^t \mu_s\, \delta s = \int_0^t \omega\, \delta s + \int_0^t \Delta\, \text{Exp}\left\{ -\lambda\, s \right\} \delta s \tag{9}$$

Using Appendix Equations (29) and (30) below the solution to Equation (9) above is...

$$\Gamma_t = \omega\, t + \frac{\Delta}{\lambda}\left(1 - \text{Exp}\left\{ -\lambda\, t \right\}\right) \tag{10}$$

We will define the variable $R_t$ to be annualized revenue at time $t$. Using Equation (10) above the equation for annualized revenue is...

$$R_t = R_0\, \text{Exp}\left\{ \Gamma_t \right\} = R_0\, \text{Exp}\left\{ \frac{\Delta}{\lambda} + \omega\, t - \frac{\Delta}{\lambda}\, \text{Exp}\left\{ -\lambda\, t \right\} \right\} \tag{11}$$

Using Appendix Equation (31) below the derivative of annualized revenue Equation (11) above with respect to time is...

$$\begin{aligned} \frac{\delta R_t}{\delta t} &= \left(\omega + \Delta\, \text{Exp}\left\{ -\lambda\, t \right\}\right) R_0\, \text{Exp}\left\{ \frac{\Delta}{\lambda} + \omega\, t - \frac{\Delta}{\lambda}\, \text{Exp}\left\{ -\lambda\, t \right\} \right\} \\ &= \omega\, R_0\, \text{Exp}\left\{ \frac{\Delta}{\lambda} + \omega\, t - \frac{\Delta}{\lambda}\, \text{Exp}\left\{ -\lambda\, t \right\} \right\} + \Delta\, R_0\, \text{Exp}\left\{ \frac{\Delta}{\lambda} + (\omega - \lambda)\, t - \frac{\Delta}{\lambda}\, \text{Exp}\left\{ -\lambda\, t \right\} \right\} \end{aligned} \tag{12}$$

## Investment

We will define the variable $A_t$ to be total assets (i.e. investment) at time $t$, and the variable $\phi$ to be the ratio of total assets to annualized revenue. Using Equation (11) above the equation for total assets is...

$$A_t = \phi\, R_t \tag{13}$$

Using Equations (12) and (13) above the equation for the derivative of total assets with respect to time is...

$$\frac{\delta A_t}{\delta t} = \phi\,\frac{\delta R_t}{\delta t} = \phi\,\omega\,R_0\,\mathrm{Exp}\left\{\frac{\Delta}{\lambda} + \omega\,t - \frac{\Delta}{\lambda}\,\mathrm{Exp}\left\{-\lambda\,t\right\}\right\} + \phi\,\Delta\,R_0\,\mathrm{Exp}\left\{\frac{\Delta}{\lambda} + (\omega - \lambda)\,t - \frac{\Delta}{\lambda}\,\mathrm{Exp}\left\{-\lambda\,t\right\}\right\} \tag{14}$$

We will define the variable $\theta$ to be the after-tax return on assets, which is defined as the ratio of annualized net income to total assets. The generalized equation for the return on assets is...

$$\theta = \text{Pre-tax revenue margin} \times \text{Annualized revenue} \times (1 - \text{Tax rate}) \div \text{Assets} \tag{15}$$

We will define the variable $\theta_t$ to be the continuous time return on assets at time $t$. The return on assets is a function of the variable $\eta$, which is the long-term sustainable rate, the variable $\psi$, which is the difference between the current unsustainable rate and the long-term sustainable rate, the variable $\lambda$, which is the rate of mean reversion, and the variable $t$, which is time in years. The equation for the return on assets is...

$$\theta_t = \eta + \psi\,\mathrm{Exp}\left\{-\lambda\,t\right\} \text{ ...where... } \eta = \theta_\infty \text{ ...and... } \psi = \theta_\infty - \theta_0 \tag{16}$$

## Cash Flow

We will define the variable $C_t$ to be the present value at time zero of annualized cash flow expected to be received at some future time $t$. The generalized equation for the present value of annualized cash flow is...

$$C_t = \Big(\text{Net Income} - \text{Investment (i.e. the change in assets)}\Big) \times \text{Discount Factor} \tag{17}$$

We will define the variable $\kappa$ to be the risk-adjusted discount rate. Using Equations (13), (14) and (16) above we can rewrite Equation (17) above as...

$$C_t = \theta_t\,A_t\,\mathrm{Exp}\left\{-\kappa\,t\right\}\delta t - \delta A_t\,\mathrm{Exp}\left\{-\kappa\,t\right\}\delta t \tag{18}$$

We will define the equation $E_1$ to be...

$$\begin{aligned} E_1 &= \mathrm{Exp}\left\{\frac{\Delta}{\lambda} + \omega\,t - \frac{\Delta}{\lambda}\,\mathrm{Exp}\left\{-\lambda\,t\right\}\right\}\mathrm{Exp}\left\{-\kappa\,t\right\} \\ &= \mathrm{Exp}\left\{\frac{\Delta}{\lambda} + (\omega - \kappa)\,t - \frac{\Delta}{\lambda}\,\mathrm{Exp}\left\{-\lambda\,t\right\}\right\} \end{aligned} \tag{19}$$

We will define the equation $E_2$ to be...

$$\begin{aligned} E_2 &= \mathrm{Exp}\left\{-\lambda\,t\right\}\mathrm{Exp}\left\{\frac{\Delta}{\lambda} + \omega\,t - \frac{\Delta}{\lambda}\,\mathrm{Exp}\left\{-\lambda\,t\right\}\right\}\mathrm{Exp}\left\{-\kappa\,t\right\}\Big\} \\ &= \mathrm{Exp}\left\{\frac{\Delta}{\lambda} + (\omega - \lambda - \kappa)\,t - \frac{\Delta}{\lambda}\,\mathrm{Exp}\left\{-\lambda\,t\right\}\right\} \end{aligned} \tag{20}$$

Using Equations (19) and (20) above we can rewrite Equation (18) above as...

$$C_t = \phi\,R_0\Big[\eta\,E_1 + \psi\,E_2 - \omega\,E_1 - \Delta\,E_2\Big] = \phi\,R_0\Big[(\eta - \omega)\,E_1 + (\psi - \Delta)\,E_2\Big] \tag{21}$$

Using Equation (21) above the equation for the present value of expected cash flow is...

$$V_0 = \int\limits_0^\infty C_t\,\delta t = \phi\,R_0\left[(\eta - \omega)\int\limits_0^\infty E_1\,\delta t + (\psi - \Delta)\int\limits_0^\infty E_2\,\delta t\right] \tag{22}$$

See Appendix Equations (36) and (41) below for the solution to each of the two integrals in Equation (22) above.

## The Answer To Our Hypothetical Problem

We will define the long-term sustainable revenue growth rate to be the expected growth rate of nominal GDP over the next ten years, which is four percent. Using the model parameters in Table 1 above the continuous time values for the long-term sustainable revenue growth rate $\omega$ and the difference between the current unsustainable rate and the long-term sustainable rate ($\Delta$) are...

$$\omega = \ln(1+0.04) = 0.0392 \text{ ...and... } \Delta = \ln(1+0.18) - \ln(1+0.04) = 0.1263 \tag{23}$$

We will define the long-term sustainable return on assets to be the cost of capital. Using the model parameters in Table 1 above the continuous time values for the long-term sustainable return on assets $\eta$ and the difference between the current unsustainable rate and the long-term sustainable rate ($\psi$) are...

$$\eta = \ln(1+0.10) = 0.0953 \text{ ...and... } \psi = \ln(1+0.15) - \ln(1+0.10) = 0.0445 \tag{24}$$

Using the model parameters in Table 1 above the value of the model parameter that represents the ratio of total assets to annualized revenue ($\phi$) is...

$$\phi = 0.8000 \tag{25}$$

Using the model parameters in Table 1 above the value of the model parameter that represents the cost of capital ($\kappa$) is...

$$\kappa = \ln(1+0.10) = 0.0953 \tag{26}$$

Using Equation (6) above and the parameters to our hypothetical problem the value of parameter $\lambda$ is...

$$\lambda = -\frac{\ln(0.50)}{5.00} = 0.1386 \tag{27}$$

Using Appendix Equations (36) and (41) below the solution to Equation (22) above, which is the answer to our hypothetical problem, is...

$$V_0 = 0.80 \times 1,000,000 \times \bigg[(0.0953 - 0.0392) \times 35.2250 + (0.0445 - 0.1263) \times 7.7310\bigg] = 1,075,000 \tag{28}$$

# References


[1] Gary Schurman, *The Incomplete Gamma Function - Part II,* December, 2017


## Appendix

**A.** The solution to the following integral is...

$$\int\limits_0^t \omega\,\delta s = \omega \int\limits_0^t \delta s = \omega \times s \bigg[_{s=0}^{s=t} = \omega\,t \tag{29}$$

**B.** The solution to the following integral is...

$$\int\limits_0^t \Delta\,\mathrm{Exp}\bigg\{-\lambda\,s\bigg\}\,\delta s = \Delta \int\limits_0^t \mathrm{Exp}\bigg\{-\lambda\,s\bigg\}\,\delta s = -\frac{\Delta}{\lambda}\,\mathrm{Exp}\bigg\{-\lambda\,s\bigg\}\bigg[_{s=0}^{s=t} = \frac{\Delta}{\lambda}\bigg(1 - \mathrm{Exp}\bigg\{-\lambda\,t\bigg\}\bigg) \tag{30}$$

**C**. The derivative of the following equation is...

$$\frac{\delta}{\delta t}\bigg[\omega\,t + \frac{\Delta}{\lambda}\bigg(1 - \mathrm{Exp}\bigg\{-\lambda\,t\bigg\}\bigg)\bigg\}\bigg] = \omega - \frac{\Delta}{\lambda} \times -\lambda \times \mathrm{Exp}\bigg\{-\lambda\,t\bigg\} = \omega + \Delta\,\mathrm{Exp}\bigg\{-\lambda\,t\bigg\} \tag{31}$$

**D**. The solution to the first integral in Equation (22) above is...

Using Equations (23) to (27) above the values for parameters $a$, $b$, $c$ and $d$ in equation $E_1$ above (Equation (19)) are...

$$a = \frac{0.1263}{0.1386} = 0.9110 \;,\; b = 0.1386 \;,\; c = 0.0392 - 0.0953 = -0.0561 \;,\; d = \frac{0.1263}{0.1386} = 0.9110 \tag{32}$$

Using Equation (32) above the solution to the first two parameter values in base Equation (2) above are...

$$\text{Exp}\Big\{d\Big\} = \text{Exp}\Big\{0.9110\Big\} = 2.4868 \text{ ...and... } a^{\frac{c}{b}}b^{-1} = 0.9110^{-\frac{0.0561}{0.1386}} \times 0.1386^{-1} = 7.4924 \tag{33}$$

Using Equation (32) above the solution to the second incomplete gamma function in base Equation (2) above where $m = 0$ is...

$$\Gamma\Bigg(-\frac{-0.0561}{0.1386}, 0.9110 \times \text{Exp}\Big\{-0.1386 \times 0\}\Bigg) = \Gamma\Bigg(0.4048, 0.9110\Bigg) = 0.3009 \tag{34}$$

Using Equation (32) above the solution to the first incomplete gamma function in base Equation (2) above where $n = \infty$ is...

$$\Gamma\Bigg(-\frac{-0.0561}{0.1386}, 0.9110 \times \text{Exp}\Big\{-0.1386 \times \infty\}\Bigg) = \Gamma\Bigg(0.4048, 0\Bigg) = 2.1912 \tag{35}$$

Using Equations (33), (34) and (35) above the value of the integral of Equation (22) above is...

$$\int\limits_m^n E_1\,\delta t = 2.4868 \times 7.4924 \times (2.1912 - 0.3009) = 35.2205 \tag{36}$$

**E**. The solution to the second integral in Equation (22) above is...

Using Equations (23) to (27) above the values for parameters $a$, $b$, $c$ and $d$ in equation $E_1$ above (Equation (19)) are...

$$a = \frac{0.1263}{0.1386} = 0.9110 \ , \ b = 0.1386 \ , \ c = 0.0392 - 0.1386 - 0.0953 = -0.1947 \ , \ d = \frac{0.1263}{0.1386} = 0.9110 \tag{37}$$

Using Equation (37) above the solution to the first two parameter values in base Equation (2) above are...

$$\text{Exp}\Big\{d\Big\} = \text{Exp}\Big\{0.9110\Big\} = 2.4868 \text{ ...and... } a^{\frac{c}{b}}b^{-1} = 0.9110^{-\frac{0.1947}{0.1386}} \times 0.1386^{-1} = 8.2244 \tag{38}$$

Using Equation (37) above the solution to the second incomplete gamma function in base Equation (2) above where $m = 0$ is...

$$\Gamma\Bigg(-\frac{-0.1947}{0.1386}, 0.9110 \times \text{Exp}\Big\{-0.1386 \times 0\}\Bigg) = \Gamma\Bigg(1.4048, 0.9110\Bigg) = 0.5090 \tag{39}$$

Using Equation (37) above the solution to the first incomplete gamma function in base Equation (2) above where $n = \infty$ is...

$$\Gamma\Bigg(-\frac{-0.1947}{0.1386}, 0.9110 \times \text{Exp}\Big\{-0.1386 \times \infty\}\Bigg) = \Gamma\Bigg(1.4048, 0\Bigg) = 0.8870 \tag{40}$$

Using Equations (38), (39) and (40) above the value of the integral of Equation (22) above is...

$$\int\limits_m^n E_1\,\delta t = 2.4868 \times 8.2244 \times (0.8870 - 0.5090) = 7.7310 \tag{41}$$